Problem 2.37

The result (2.57) for the velocity of a falling object was found by integrating Equation (2.55) and the quickest way to do this is to use the integral $\int du/(1-u^2) = \operatorname{arctanh} u$. Here is another way to do it: Integrate (2.55) using the method of "partial fractions," writing

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right),$$

which lets you do the integral in terms of natural logs. Solve the resulting equation to give v as a function of t and show that your answer agrees with (2.57).

Solution

Begin with Equation (2.55) on page 60.

$$\frac{dv}{1 - v^2/v_{\text{ter}}^2} = g\,dt \tag{2.55}$$

Integrate both sides definitely, assuming the object's initial velocity is zero.

$$\int_0^v \frac{dv'}{1 - v'^2/v_{\text{ter}}^2} = \int_0^t g\,dt'$$

$$= gt$$

Make the following substitution in the integral on the left.

$$u = \frac{v'}{v_{\text{ter}}}$$

$$du = \frac{dv'}{v_{\text{ter}}} \quad \rightarrow \quad v_{\text{ter}}\,du = dv'$$

Consequently,

$$\int_0^{v/v_{\text{ter}}} \frac{v_{\text{ter}}\,du}{1-u^2} = gt$$

$$v_{\text{ter}} \int_0^{v/v_{\text{ter}}} \frac{du}{1-u^2} = gt$$

$$v_{\text{ter}} \int_0^{v/v_{\text{ter}}} \frac{1}{(1+u)(1-u)}\,du = gt. \tag{1}$$

Use partial fraction decomposition to write the integrand as two terms.

$$\frac{1}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u}$$

Multiply both sides by the least common denominator, $(1+u)(1-u)$.

$$1 = A(1-u) + B(1+u)$$

The constants, A and B, hold for all values of u, so set $u = 1$ and $u = -1$ in this equation to determine A and B.

$$u = 1: \qquad 1 = A(0) + B(2) \qquad \rightarrow \qquad 1 = 2B \qquad \rightarrow \qquad B = \frac{1}{2}$$

$$u = -1: \qquad 1 = A(2) + B(0) \qquad \rightarrow \qquad 1 = 2A \qquad \rightarrow \qquad A = \frac{1}{2}$$

Therefore,

$$\frac{1}{(1+u)(1-u)} = \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u},$$

and equation (1) becomes

$$v_{\text{ter}} \int_0^{v/v_{\text{ter}}} \left(\frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u} \right) du = gt$$

$$\frac{v_{\text{ter}}}{2} \int_0^{v/v_{\text{ter}}} \left(\frac{1}{u+1} - \frac{1}{u-1} \right) du = gt$$

$$\frac{v_{\text{ter}}}{2} \left(\int_0^{v/v_{\text{ter}}} \frac{du}{u+1} - \int_0^{v/v_{\text{ter}}} \frac{du}{u-1} \right) = gt$$

$$\frac{v_{\text{ter}}}{2} \left(\ln|u+1| \Big|_0^{v/v_{\text{ter}}} - \ln|u-1| \Big|_0^{v/v_{\text{ter}}} \right) = gt$$

$$\frac{v_{\text{ter}}}{2} \left[\left(\ln\left| \frac{v}{v_{\text{ter}}} + 1 \right| - \ln 1 \right) - \left(\ln\left| \frac{v}{v_{\text{ter}}} - 1 \right| - \ln 1 \right) \right] = gt$$

$$\frac{v_{\text{ter}}}{2} \left(\ln\left| \frac{v}{v_{\text{ter}}} + 1 \right| - \ln\left| \frac{v}{v_{\text{ter}}} - 1 \right| \right) = gt$$

$$\frac{v_{\text{ter}}}{2} \ln\left| \frac{\frac{v}{v_{\text{ter}}} + 1}{\frac{v}{v_{\text{ter}}} - 1} \right| = gt$$

$$\frac{1}{2} \ln\left| \frac{\frac{v}{v_{\text{ter}}} + 1}{\frac{v}{v_{\text{ter}}} - 1} \right| = \frac{gt}{v_{\text{ter}}}$$

$$\frac{1}{2} \ln\left(\frac{1 + \frac{v}{v_{\text{ter}}}}{1 - \frac{v}{v_{\text{ter}}}} \right) = \frac{gt}{v_{\text{ter}}}. \qquad (2)$$

The goal now is to show that

$$\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z} \right),$$

or

$$z = \tanh\left[\frac{1}{2} \ln\left(\frac{1+z}{1-z} \right) \right].$$

Recall that

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{\frac{e^z - e^{-z}}{2}}{\frac{e^z + e^{-z}}{2}} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}.$$

As a result,

$$\begin{aligned}
\tanh\left[\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)\right] &= \frac{\exp\left[\ln\left(\frac{1+z}{1-z}\right)\right] - 1}{\exp\left[\ln\left(\frac{1+z}{1-z}\right)\right] + 1} \\
&= \frac{\frac{1+z}{1-z} - 1}{\frac{1+z}{1-z} + 1} \\
&= \frac{(1+z) - (1-z)}{(1+z) + (1-z)} \\
&= \frac{2z}{2} \\
&= z.
\end{aligned}$$

Equation (2) then becomes

$$\tanh^{-1}\left(\frac{v}{v_{\text{ter}}}\right) = \frac{gt}{v_{\text{ter}}},$$

which means

$$\frac{v}{v_{\text{ter}}} = \tanh\left(\frac{gt}{v_{\text{ter}}}\right).$$

Multiply both sides by v_{ter} to get Equation (2.57) on page 61.

$$v = v_{\text{ter}} \tanh\left(\frac{gt}{v_{\text{ter}}}\right) \tag{2.57}$$